Consider the sequence defined recursively by $a_0 = 2, \quad a_n = 5a_{n-1} + 8$ for $n \in \mathbf{Z}^+$. **SCORE: _____ / 15 PTS**

[a] Using iteration, guess an explicit formula for the sequence (ie. a general formula for a_n).

1 EACH

$$a_1 = 5a_0 + 8 = \boxed{5 \cdot 2 + 8}$$

$$a_2 = 5a_1 + 8 = 5 \cdot (5 \cdot 2 + 8) + 8 = \boxed{5^2 \cdot 2 + 5 \cdot 8 + 8}$$

$$a_3 = 5a_2 + 8 = 5 \cdot (5^2 \cdot 2 + 5 \cdot 8 + 8) + 8 = \boxed{5^3 \cdot 2 + 5^2 \cdot 8 + 5 \cdot 8 + 8}$$

$$a_n = 5^n \cdot 2 + (5^{n-1} \cdot 8 + \cdots + 5 \cdot 8 + 8) = 2 \cdot 5^n + \frac{8(5^n - 1)}{5 - 1} = \boxed{4 \cdot 5^n - 2}$$

[b] **Using mathematical induction**, prove that your formula in [a] is correct.

Proof by Mathematical Induction:

Basis step: $\boxed{a_0 = 4 \cdot 5^0 - 2} = \boxed{4 - 2 = 2}$

Inductive step: $\boxed{\text{Assume that } a_k = 4 \cdot 5^k - 2}$ $\boxed{\text{for some particular but arbitrary integer } k \geq 0}$

[Prove that $a_{k+1} = 4 \cdot 5^{k+1} - 2$]

$k \geq 0$, so $\boxed{k + 1 \geq 1}$

So, $\boxed{a_{k+1} = 5a_k + 8}$ $\boxed{= 5(4 \cdot 5^k - 2) + 8}$ $\boxed{= 4 \cdot 5^{k+1} - 10 + 8}$ $\boxed{= 4 \cdot 5^{k+1} - 2}$

$\boxed{\text{By mathematical induction}}$ $\boxed{a_n = 4 \cdot 5^n - 2 \text{ for all non-negative integers } n}$

Simplify the following expressions. **SCORE: _____ / 7 PTS**

[a] $\dfrac{(3n-2)!}{(3n+1)!}$ (assuming $n \in \mathbf{Z}^+$)

$$= \frac{(3n-2)!}{(3n+1)(3n)(3n-1)(3n-2)!}$$ 2

$$= \frac{1}{(3n+1)(3n)(3n-1)}$$ 1

[b] $\dbinom{9}{6}$

$$= \frac{9!}{6!\,3!}$$ 1

$$= \frac{9\cdot 8\cdot 7\cdot 6!}{6!\,3\cdot 2\cdot 1}$$ 1

$$= 3\cdot 4\cdot 7$$ 1

$$= 84$$ 1

One of the following statements is true and one is false. **SCORE: _____ / 13 PTS**

State clearly which statement is false, show that it is false, then write a **formal proof** for the true statement **using mathematical induction**.

[a] $(n^3 + 2n - 1) \bmod 3 = 2$ for all positive integers n

[b] $3 \mid (n^3 - n^2)$ for all non-negative integers n

[a] is true.

Proof by Mathematical Induction:

Basis step: $(1^3 + 2 \cdot 1 - 1) \bmod 3 = 2$ $\frac{1}{2}$ since $2 = 3 \cdot 0 + 2$ $\frac{1}{2}$

Inductive step: 1 Assume that $(k^3 + 2k - 1) \bmod 3 = 2$ for some particular but arbitrary $k \in \mathbf{Z}^+$ $\frac{1}{2}$

[Prove that $((k+1)^3 + 2(k+1) - 1) \bmod 3 = 2$]

1 $k^3 + 2k - 1 = 3q + 2$ for some $q \in \mathbf{Z}$ by definition of mod 1

$\frac{1}{2}$ $(k+1)^3 + 2(k+1) - 1 = k^3 + 3k^2 + 3k + 1 + 2k + 2 - 1 = k^3 + 2k - 1 + 3(k^2 + k + 1)$ 1

1 $= 3(q + k^2 + k + 1) + 2$ and $q + k^2 + k + 1 \in \mathbf{Z}$ by closure of $\mathbf{Z}$ under + and · 1

1 So, $((k+1)^3 + 2(k+1) - 1) \bmod 3 = 2$ by definition of mod 1

$\frac{1}{2}$ By mathematical induction, $(n^3 + 2n - 1) \bmod 3 = 2$ for all positive integers n $\frac{1}{2}$

[b] is false. If $n = 2$, $n^3 - n^2 = 4$ and $3 \nmid 4$. 2 TO GET ANY POINTS, MUST HAVE VALID COUNTEREXAMPLE